# Self-enforcing Debt Limits and Costly Default in General Equilibrium

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# This paper

Q: How much borrowing is sustainable, if borrowers cannot commit?

- General equilibrium model of competitive risk-sharing
  - **Defaulters suffer endowment loss**
    (Eaton Gersovitz, ...)
  - **Multilateral** lack of commitment
  - Interest rates & debt limits are **endogenous**
    (Alvarez Jermann, Kehoe Levine, Kocherlakota, ...)
  - Defaulters **excluded from borrowing, but can save**
    (Bulow Rogoff, Hellwig Lorenzoni)

# Results & contributions

1. **Max debt limits = PV of default cost**
   - Coro 1: Bulow Rogoff in g.e. w/ multilateral lack of commitment
   - Coro 2: Limiting case of debt as Ponzi schemes (Hellwig Lorenzoni)
2. **"Institutional mapping"**: Payoff-equivalence between model with "implicit insitutional" and model with "explicit institutions"
   - Public debt backed by taxes
   - Consumer debt backed by pledgeable income (Gottardi Kubler)
   - Consumer debt collateralized by assets (Geanakoplos et al.)

# Outline

1. Environment
2. Main result
3. Mapping to models with explicit institutions

# Environment

# Environment

- Underlying stochastic process: event tree of all possible states $s^t$
- Finite set $I$ of types. Stochastic endowment $\{y^i(s^t) > 0\}$ of perishable good
- $U(c) := E_0 \sum_{t\geq 0} \beta^t u(c(s^t))$, Inada conditions
- Trade one-period state-contingent debt. Cannot commit to repay.
- Subject to **finite** non-negative debt limits $D^i(s^t)$

# Repay value

- $\forall s^t$, given inherited $a$ & limits $D^i$,

$$V^i(D^i, a|s^t) := \sup\{U(c^i|s^t) \ : \ (c^i, a^i) \in B^i(D^i, a|s^t)\}$$

- $B^i(D^i, a|s^t) := \{(c^i, a^i)| \qquad a^i(s^t) = a,$

$$\begin{aligned} c^i(s^{t'}) + \sum_{s^{t'+1} \succ s^{t'}} q(s^{t'+1}) a^i(s^{t'+1}) &\leq y^i(s^{t'}) + a^i(s^{t'}), \\ a^i(s^{t'+1}) &\geq -D^i(s^{t'+1}) \quad \forall s^{t'} \succeq s^t\} \end{aligned}$$

- For $B^i \neq \emptyset$, assume WLOG debt limits are *consistent*:

$$D^i(s^t) \leq y^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1})$$

# Default value

- Assume defaulters
  - cannot borrow, but can save
  - lose fraction $\tau \geq 0$ of endowment

$$V_d^i(0,0|s^t) := \sup\{U(c^i|s^t) \ : \ (c^i, a^i) \in B_d^i(0,0|s^t)\}$$

- $B_d^i(0,0|s^t) := \{(c^i, a^i)| \qquad a^i(s^t) = 0,$

$$c^i(s^{t'}) + \sum_{s^{t'+1} \succ s^{t'}} q(s^{t'+1}) a^i(s^{t'+1}) \leq \underbrace{(1 - \tau^i(s^{t'})) y^i(s^{t'})}_{=: y_d^i(s^{t'})} + a^i(s^{t'}),$$

$$a^i(s^{t'+1}) \geq 0 \quad \forall s^{t'} \succeq s^t\}$$

# Boundedness

- To guarantee finite continuation value, assume

$$
\begin{aligned}
U((1-\tau^i)y^i|s^t) &> -\infty, \\
U(\sum_i y^i|s^t) &< \infty, \quad \forall s^t
\end{aligned}
$$

- True if either
  - $u$ is bounded, or
  - $(1-\tau^i)y^i$ uniformly bounded away from 0
    & $y^i$ uniformly bounded from above

# Non-negligible loss

Assume *aggregate* endowment loss is *non-negligible* (with respect to aggregate endowments): $\exists \varepsilon > 0$ s.t.

$$\frac{\sum_{i \in I} \tau^i(s^t) y^i(s^t)}{\sum_{i \in I} y^i(s^t)} \geq \varepsilon, \quad \forall s^t$$

- E.g. 1: $\tau^i \geq \varepsilon, \quad \forall i$
- E.g. 2:
  - Committed types: $\tau^i = 1, \quad \forall i \in I^c$
  - Non-committed types: $\tau^i = 0, \quad \forall i \in I^{nc}$
  - Committed types' endowments are non-negligible: $\frac{\sum_{i \in I^c} y^i}{\sum_{i \in I} y^i} > \varepsilon$

# Definitions: Self-enforcing debt limits (Alvarez Jermann)

- Maximum sustainable debt captured by "not-too-tight debt limits"
- $D^i$ is *self-enforcing* (or sustainable) if $\forall s^t$

$$V^i(D^i, -D^i(s^t)|s^t) \geq V^i_d(0, 0|s^t)$$

- $D^i$ is *not-too-tight* (or maximally sustainable) if '$=$' $\forall s^t$
  - These debt limits prevent default, but allow as much borrowing as possible
  - These debt limits arise endogenously in competitive market

## Definition

*For initial market-clearing $\{a^i(s^0)\}_{i \in I}$, a competitive equilibrium with self-enforcing debt $(q, (c^i, a^i, D^i)_{i \in I})$ satisfies*

1. *individual optimization (taking prices & debt limits as given)*
2. *debt market clears* $\sum_{i \in I} a^i(s^t) = 0, \quad \forall s^t$
3. *debt limits $D^i$ are not-too-tight.*

# Result I:

$$D = \mathrm{PV}(\tau y)$$

# Notations

- Present value & wealth:

$$\begin{aligned} \mathrm{PV}(x|s^t) &:= \frac{1}{p(s^t)} \sum_{s^{t+\tau} \succeq s^t} p(s^{t+\tau}) x(s^{t+\tau}) \\ W^i(s^t) &:= \mathrm{PV}(y^i|s^t) \end{aligned}$$

  - Date-0 price of consumption good:

$$\begin{aligned} p(s^0) &:= 1 \\ p(s^{t+1}) &:= q(s^{t+1}) p(s^t) \end{aligned}$$

  - Deterministic special case:

$$\begin{aligned} \mathrm{PV}_t(x) &:= \sum_{t+\tau \geq t} \frac{x_{t+\tau}}{\Pi_{\tau \geq 0}(1 + r_{t+\tau})} \\ 1 + r_t &:= \frac{1}{q_t} \end{aligned}$$

### Theorem 1

*Assume non-negligible $\tau$.*
*Equilibrium debt limits must = present value of endowment loss:*

$$D^i(s^t) = \mathrm{PV}(\tau^i y^i | s^t), \quad \forall s^t, i$$

# Example

- $(y_t^1)_{t\geq 0} = (y_H, y_L, y_H, y_L, \dots)$
- $(y_t^2)_{t\geq 0} = (y_L, y_H, y_L, y_H, \dots)$
- $u = \log$; identical loss $\tau$
- Stationary equilibrium:

$$\begin{aligned} V(D^i, -D^i) &= V_d^i(0,0), \quad \forall i \\ foc_H: \quad q &= \beta \frac{u'(c_L)}{u'(c_H)} \\ foc_L: \quad q &\geq \beta \frac{u'(c_H)}{u'(c_L)} \end{aligned}$$

- What is $D^i$?

# Example (cont.)

- If $0 < \tau < \tau^*$, then unique stationary equilibrium:

$$\begin{aligned} D^i &= \mathrm{PV}(\tau y^i) = \begin{cases} \tau \frac{y_H + q y_L}{1-q^2} =: d_H \\ \tau \frac{y_L + q y_H}{1-q^2} =: d_L \end{cases} \\ foc_H: \quad q &= \beta \frac{u'(c_L)}{u'(c_H)} = \frac{u'(y_L + d_H + q d_H)}{u'(y_H - d_H - q d_H)} \\ foc_L: \quad q &> \beta \frac{u'(c_H)}{u'(c_L)} \end{aligned}$$

- If $\tau \geq \tau^*$, then first best: $q = \beta$, $c_L = c_H$, $D^i$ never binds

# Example (cont.)

- If $\tau = 0$. Let $\frac{1}{q_{aut}} := \frac{u'(y_H)}{\beta u'(y_L)}$
  - If $\frac{1}{q_{aut}} \geq 1$, then unique stationary equilibrium is no trade
  - Else, multiple stationary equilibria. One with no trade. One with bubble:

$$\begin{aligned} q &= 1 \\ D^i &= d \text{ that solves } 1 = \beta \frac{u'(y_L + 2d)}{u'(y_H - 2d)} \end{aligned}$$

  - Bubbly equilibrium is "stable"

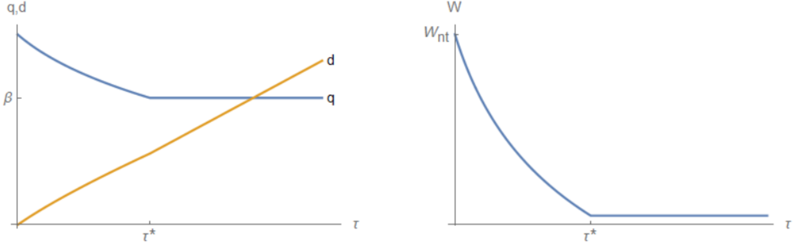


Figure: Example with no bubble ($1/q_{aut} \geq 1$)

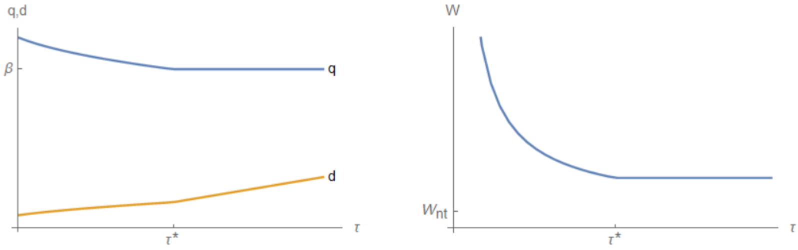


Figure: Example with bubble ($1/q_{aut} < 1$)

# Steps of proof

To show $D^i = \text{PV}(\tau^i y^i), \quad \forall i$

1. Show $D^i \geq \text{PV}(\tau^i y^i)$
   - Corollary: $W^i = \text{PV}(y^i)$ finite
   - Corollary: "overturn" Hellwig Lorenzoni
2. Show $D^i \leq \text{PV}(\tau^i y^i)$
   - Generalize Bulow Rogoff to general equilibrium environment

# Step 1: Lower bound on debt limits

**Proposition 1**

*Not-too-tight* $D^i(s^t) \geq \mathrm{PV}(\tau^i y^i | s^t), \quad \forall i, s^t$

- Note: hold for any $\tau \geq 0$
- Equivalent to $V^i(D^i, -\mathrm{PV}(\tau^i y^i | s^t) | s^t) \geq V_d^i(0, 0 | s^t)$
- Straightforward if default leads to autarky (Kehoe Levine, Alvarez Jermann). But not here, as defaulter can still save

# Sketch of proof

1. For each finite $D$, show $\exists \underline{D} \geq 0$

$$\underline{D}(s^t) = \tau(s^t)y(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), \underline{D}(s^{t+1})\}$$

2. If $D$ not-too-tight, then $D \geq \underline{D}$, i.e.,

$$V(D, -\underline{D}(s^t)|s^t) \geq V_d(0,0|s^t)$$

3. Thus $$D(s^t) \geq \underline{D}(s^t) = \underbrace{\tau(s^t)y(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\underline{D}(s^{t+1})}_{\to \text{PV}(\tau y|s^t)}$$

# Finite wealth

## Corollary 2

*Assume non-negligible $\tau$. Equilibrium interest rates must be high:*

$$\sum_{i \in I} W^i(s^0) < \infty$$

- Implication: bubbles cannot exist (Santos Woodford 1997)
- Contrast to Hellwig Lorenzoni (2009), where $\tau \equiv 0$ and $W = \infty$

## Proof.

From lemma:

$$\sum_{i \in I} D^i(s^0) \geq \sum_{i \in I} \mathrm{PV}(\tau^i y^i | s^0)$$

Since the aggregate output loss is *non-negligible*

$$\underbrace{\sum_{i \in I} D^i(s^0)}_{\text{finite}} \geq \varepsilon \sum_{i \in I} \underbrace{\mathrm{PV}(y^i | s^0)}_{W^i(s^0)}$$

□

# Step 2: Upper bound on debt limits

**Proposition 2**

*Assume non-negligible $\tau$. Then*

$$D^i(s^t) \leq \mathrm{PV}(\tau^i y^i | s^t), \quad \forall i, s^t$$

# Natural debt limits

## Lemma 3

*Assume non-negligible $\tau$. Equilibrium debt limits are bounded by natural debt limits:*

$$D^i(s^t) \leq W^i(s^t) \quad \forall s^t, i$$

# Sketch of proof

- Consistency $D^i(s^t) \leq y^i(s^t) + \sum_{s^{t+1} \succ s^t} D^i(s^{t+1})$ implies
$$D^i(s^t) \leq W^i(s^t) + M^i(s^t)$$
- Where
$$M^i(s^t) := \lim_{\tau \to \infty} \sum_{s^\tau \in S^\tau(s^t)} \frac{p(s^\tau)}{p(s^t)} D^i(s^\tau) \geq 0$$
- NTS $M^i = 0$
  - Finite PV of consumption & Inada condition $\Rightarrow$ market TVC
  - Consolidating budget constraints:
$$\begin{aligned} & PV(c^i|s^t) + \overbrace{\lim_{\tau \to \infty} \sum_{s^\tau \in S^\tau(s^t)} \frac{p(s^\tau)}{p(s^t)} [a^i(s^\tau) + D^i(s^\tau)]}^{=0 \text{ (TVC)}} \\ = \quad & \text{PV}(y^i|s^t) + M^i(s^t) + a^i(s^t) \end{aligned}$$
  - Aggregate over $i$ & use market clearing, get $\sum_{i \in I} M^i = 0$

$\Rightarrow$ $M^i = 0$

# Generalization of Bulow Rogoff

## Lemma 4

*Fix arbitrary i & self-enforcing $D^i$. If*

1. *Interest rate so high that wealth finite:* $W^i(s^0) < \infty$
2. $D^i$ *bounded by natural debt limit:* $D^i(s^t) \leq W^i(s^t), \quad \forall s^t$

*then*

$$D^i(s^t) \leq \text{PV}(\tau^i y^i | s^t), \quad \forall s^t$$

- Special case: no trade theorem $\tau^i \equiv 0 \Rightarrow D^i \equiv 0$
- We showed: non-negligible $\tau \Rightarrow$ 1 & 2 **endogenously** $\Rightarrow D \leq \text{PV}$

# Take-aways

Forces that pin down debt limits in competitive equilibrium:

- Non-negligible loss $\Rightarrow$ high interest rates, finite aggregate wealth
- Threat of default + high interest rates $\Rightarrow$ self-enforcing debt limits $\leq$ PV of loss
- Competition $\Rightarrow$ not-too-tight debt limits $\geq$ PV of loss
- Thus $D =$ PV of loss
  - Similar to competitive pricing of Lucas tree at PV of dividends

# Equivalence results:

# Model with backed public debt

# Environment

- Agents cannot issue private debt: $D^i \equiv 0$
- But can buy public debt, issued by a fiscal authority with tax $\tau$
- Private budget set: $\hat{B}^i(a|s^\tau) := \{(c^i, \hat{a}^i)| \quad \hat{a}^i(s^\tau) = a,$

$$\begin{aligned} c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\hat{a}^i(s^{t+1}) &\leq (1 - \tau^i(s^t))y^i(s^t) + \hat{a}^i(s^t), \\ \hat{a}^i(s^{t+1}) &\geq 0 \quad \forall s^t \succeq s^\tau\} \end{aligned}$$

  - Note: $\hat{B}^i(a|s^t) = B^i_d(0, a|s^t)$

# Environment (cont.)

- Gov budget constraint:

$$d(s^t) = \underbrace{\sum_{i \in I} \tau^i(s^t) y^i(s^t)}_{\text{tax}} + \underbrace{\sum_{s^{t+1} \succ s^t} q(s^{t+1}) d(s^{t+1})}_{\text{roll over}}, \quad \forall s^t$$

- Equilibrium: public debt market clears:

$$\sum_{i \in I} a^i(s^t) = d(s^t), \quad \forall s^t$$

- Assume $\tau$ non-negligible

# Finite wealth

**Lemma 5 (Finite wealth)**

$$\sum_{i \in I} W^i(s^0) < \infty$$

**Proposition 3 (Debt = PV taxes)**

$$d(s^t) = \text{PV}(\sum_{i \in I} \tau^i y^i | s^t), \quad \forall s^t$$

# Payoff & price equivalence

**Proposition 4**

$(q, d, (c^i, \hat{a}^i)_{i \in I})$ *competitive equilibrium with public debt backed by tax* $\tau$

$\Longleftrightarrow$

$(q, (c^i, a^i, D^i)_{i \in I})$ *competitive equilibrium with self-enforcing private debt and endowment loss* $\tau$*, where*

$$
\begin{aligned}
D^i &= \mathrm{PV}(\tau^i y^i) \\
a^i &= \hat{a}^i - D^i
\end{aligned}
$$

- Mapping of private liquidity (private individuals' debt issuance) to public liquidity (public debt issuance) (Holmstrom Tirole)

# Equivalence results:

# Constrained Arrow Debreu model

# AD with limited pledgeability (Gottardi Kubler)

- Each consumer can sell a fraction $\tau^i$ of endowments in advance (i.e., fraction $\tau^i$ of income pledgeable)

- A-D equilibrium w. limited pledgeability: $(p, (c^i)_{i \in I})$ s.t.
  - Wealth is finite: $\text{PV}(y^i|s^0) < \infty$, $\forall i$
  - Date-0 budget constraint:

$$\text{PV}(c^i|s^0) \leq a^i(s^0) + \text{PV}(y^i|s^0)$$

  - Limited pledgeability:

$$\text{PV}(c^i|s^t) \geq \underbrace{\text{PV}((1-\tau^i)y^i|s^t)}_{\text{non-pledgeable endowment}}, \forall s^t$$

  - Market clears: $\sum_{i \in I} c^i(s^t) = \sum_{i \in I} y^i(s^t)$, $\forall s^t$

# Payoff & price equivalence

## Proposition 5

$(p, (c^i)_{i \in I})$ *is AD equilibrium w. limited pledgeability*

$\Longleftrightarrow$

$(q, (c^i, a^i, D^i)_{i \in I})$ *is competitive equilibrium w. self-enforcing debt, where*

$$\begin{aligned} D^i(s^t) &= \mathrm{PV}(\tau^i y^i | s^t) \\ a^i(s^t) &= \mathrm{PV}(c^i - (1-\tau^i) y^i | s^t), \quad \forall s^t \end{aligned}$$

# Relationship to Collateral equilibrium model

GK showed: consumption allocations of constrained A-D model coincide with those in collateral equilibrium model (Geanakoplos 1997, Geanakoplos Zame 2002, 2009)

- Agents sequentially trade state-contingent securities
- & trade shares of a *collateralizable* Lucas tree (but cannot short-sell)
- Defaulters lose all collateral, but no other punishment

# Conclusion

- General equilibrium with limited commitment and endowment loss
- Show: Maximal sustainable debt = PV of default cost
- Show: Environment with "implicit insitutional" can be mapped to environments with "explicit institutions"
  - Public debt backed by taxes
  - Arrow-Debreu with limited pledgeability
  - Debt collateralized by assets